UNIVERSITY OF OXFORD

Imperial College London

WARWICK
THE UNIVERSITY OF WARWICK

MATHEMATICS ADMISSIONS TEST

For candidates applying for Mathematics, Computer Science or one of their joint degrees at OXFORD UNIVERSITY and/or IMPERIAL COLLEGE LONDON and/or UNIVERSITY OF WARWICK

Thursday 2 November 2017

Time Allowed: 2½ hours

2184051577

Please complete the following details in BLOCK CAPITALS. You must use a pen.

Surname	
Other names	

Candidate Number	**M**					

This paper contains 7 questions of which you should attempt 5. There are directions throughout the paper as to which questions are appropriate for your course.

A: Oxford Applicants: if you are applying to Oxford for the degree course:

- Mathematics *or* Mathematics & Philosophy *or* Mathematics & Statistics, you should attempt Questions **1,2,3,4,5.**
- Mathematics & Computer Science, you should attempt Questions **1,2,3,5,6.**
- Computer Science *or* Computer Science & Philosophy, you should attempt **1,2,5,6,7.**

Directions under A take priority over any directions in B which are relevant to you.

B: Imperial or Warwick Applicants: if you are applying to the University of Warwick for Mathematics BSc, Master of Mathematics, or if you are applying to Imperial College for any of the Mathematics courses**:** Mathematics, Mathematics (Pure Mathematics), Mathematics with a Year in Europe, Mathematics with Applied Mathematics/Mathematical Physics, Mathematics with Mathematical Computation, Mathematics with Statistics, Mathematics with Statistics for Finance, Mathematics Optimisation and Statistics, you should attempt Questions **1,2,3,4,5.**

Further credit cannot be obtained by attempting extra questions. **Calculators are not permitted.**

Question 1 is a multiple choice question with ten parts. Marks are given solely for correct answers but any rough working should be shown in the space between parts. Answer Question 1 on the grid on Page 2. Each part is worth 4 marks.

Answers to questions 2-7 should be written in the space provided, continuing on to the blank pages at the end of this booklet if necessary. Each of Questions 2-7 is worth 15 marks.

MATHEMATICS ADMISSIONS TEST

Thursday 2 November 2017

Time Allowed: 2½ hours

Please complete these details below in block capitals.

Centre Number														
Candidate Number	M													
UCAS Number (if known)				–				–						

	d	d		m	m		y	y
Date of Birth			–			–		

Please tick the appropriate box:

☐ I have attempted Questions **1,2,3,4,5**

☐ I have attempted Questions **1,2,3,5,6**

☐ I have attempted Questions **1,2,5,6,7**

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Q1	Q2	Q3	Q4	Q5	Q6	Q7

1. For **ALL APPLICANTS.**

For each part of the question on pages 3-7 you will be given **five** possible answers, just one of which is correct. Indicate for each part **A-J** which answer (a), (b), (c), (d), or (e) you think is correct with a tick (✓) in the corresponding column in the table below. *Please show any rough working in the space provided between the parts.*

	(a)	(b)	(c)	(d)	(e)
A					
B					
C					
D					
E					
F					
G					
H					
I					
J					

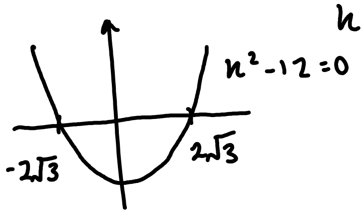
A. Let

$$f(x) = 2x^3 - kx^2 + 2x - k.$$

For what values of the real number k does the graph $y = f(x)$ have two distinct real stationary points?

(a) $-2\sqrt{3} < k < 2\sqrt{3}$
(b) $k < -2\sqrt{3}$ or $2\sqrt{3} < k$ ✓
(c) $k < -\sqrt{21} - 3$ or $\sqrt{21} - 3 < k$
(d) $-\sqrt{21} - 3 < k < \sqrt{21} - 3$
(e) all values of k.

$f(x) = 2x^3 - kx^2 + 2x - k$
$f'(x) = 6x^2 - 2kx + 2$
As there are two distinct real turning points, $f'(x)$ has two distinct solutions, hence its discriminant is positive:
$4k^2 - 4(6)(2) > 0$, $k^2 - 12 > 0$
$k^2 > 12$
$k > \pm\sqrt{12}$ $(= \pm 2\sqrt{3})$



From sketch of $k^2 - 12 = 0$, $k^2 - 12 > 0$ satisfied when $k > 2\sqrt{3}$ or $k < -2\sqrt{3}$

B. The minimum value achieved by the function

$$f(x) = 9\cos^4 x - 12\cos^2 x + 7$$

equals

(a) 3 ✓ (b) 4 (c) 5 (d) 6 (e) 7.

$9\cos^4 x - 12\cos^2 x + 7$
let $\cos^2 x = k$, a constant where $0 \leq k \leq 1$

$f(k) = 9k^2 - 12k + 7$
$f'(k) = 18k - 12 = 0$ at minimum, $k = \frac{12}{18} = \frac{2}{3}$
$f(\frac{2}{3}) = 9 \times (\frac{2}{3})^2 - 12 \times (\frac{2}{3}) + 7$
$= 4 - 8 + 7$
$= 3$

C. A sequence (a_n) has the property that

$$a_{n+1} = \frac{a_n}{a_{n-1}}$$

for every $n \geqslant 2$. Given that $a_1 = 2$ and $a_2 = 6$, what is a_{2017}?

(a) $\frac{1}{6}$ (b) $\frac{1}{3}$ (c) $\frac{1}{2}$ ✓(d) 2 (e) 3.

Calculating the first few terms of the sequence gives :

$\underbrace{2, 6, 3, \tfrac{1}{2}, \tfrac{1}{6}, \tfrac{1}{3}}_{a_1,\ a_2\ \dots\ a_6}, \underbrace{2, 6, 3, \tfrac{1}{2}, \tfrac{1}{6}, \tfrac{1}{3}}_{\text{sequence repeats every 6 terms}}, \dots$

$2017 \div 6 = 336$ r 1

$$\begin{array}{r} 336 \text{ r}1 \\ 6\,\overline{)2017} \\ \underline{18} \\ 21 \\ \underline{18} \\ 37 \\ \underline{36} \\ 1 \end{array}$$

$2017 = 1$ modulo 6

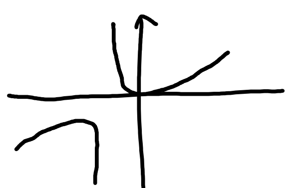
corresponds to value $a_1, a_7 \dots a_{6n+1}$

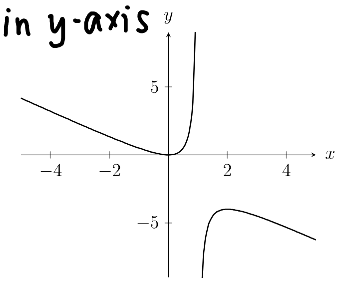
$a_1 = 2$

D. The diagram below shows the graph of $y = f(x)$.

$f(-x)$ reflects graph in y-axis

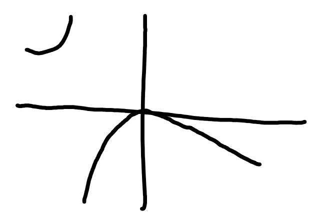
$f(-x)$



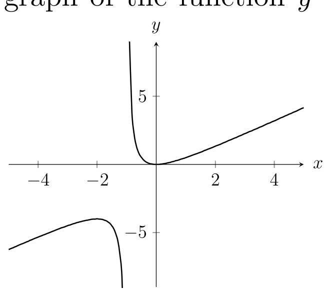


$-f(x)$ reflects graph in x axis

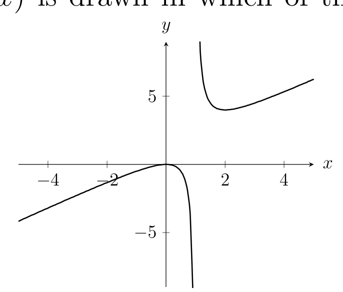
$-f(-x)$



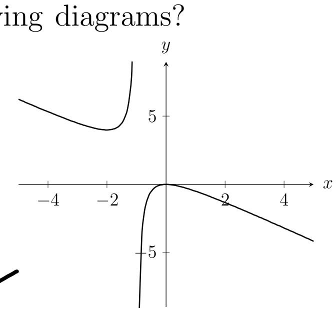
The graph of the function $y = -f(-x)$ is drawn in which of the following diagrams?



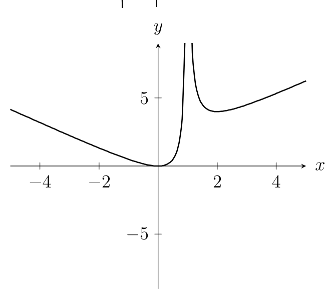
(a)



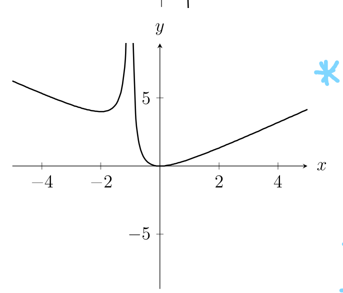
(b)



✓(c)



(d)



(e)

* here, order of reflection didn't matter, however it is good practice in other potential MAT questions to follow the correct order of transformations.

E. Let a and b be *positive integers* such that $a + b = 20$. What is the maximum value that a^2b can take?

(a) 1000 (b) 1152 (c) 1176 (d) 1183 (e) 1196.

To maximise a^2b, it can be seen that 'a' has twice the effect of 'b' as 'a' is squared.

so let $a = 2b$

$a + b = 3b = 20$

$b = \frac{20}{3}$ $a = \frac{40}{3}$

since a and b are integers, closest solutions are $a = 13$, $b = 7$

$a^2b = (13)^2(7)$

$= 1183$

Alternative solution:

$b = 20 - a$

$a^2b = a^2(20 - a)$

$= 20a^2 - a^3$

differentiate for maximum

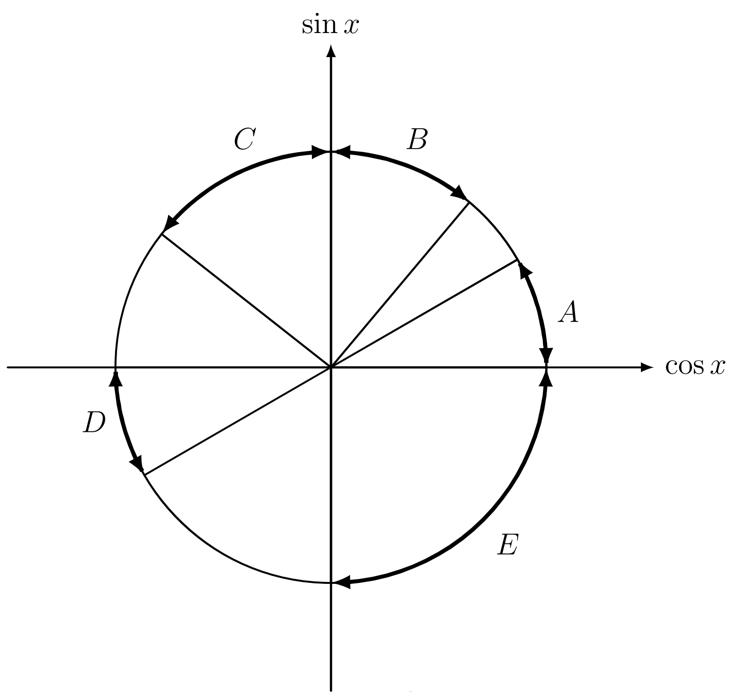
$40a - 3a^2 = 0$

$a(40 - 3a) = 0$

$a = \frac{40}{3}$, $b = \frac{20}{3}$

F. The picture below shows the unit circle, where each point has coordinates $(\cos x, \sin x)$ for some x. Which of the marked arcs corresponds to

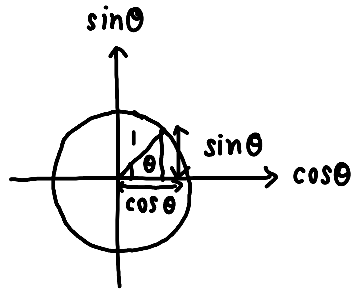
$$\tan x < \cos x < \sin x\ ?$$



(a) A (b) B (c) C (d) D (e) E.

Turn over

1F. On a unit circle:



$\sin\theta$ = height

$\cos\theta$ = base

$\tan\theta = \dfrac{\sin\theta}{\cos\theta}$ = gradient

we want

(*) $\tan\theta < \cos\theta < \sin\theta$

$\dfrac{\sin\theta}{\cos\theta} < \cos\theta < \sin\theta$

By inspection, A and E can be ruled out as $\cos\theta > \sin\theta$ in those areas.

The difference between B, C, and D are whether $\cos\theta$ and $\sin\theta$ are +ve/-ve

If $\sin\theta > \cos\theta > 0$, $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$

then $\dfrac{\sin\theta}{\cos\theta} > \dfrac{\cos\theta}{\cos\theta}$, so $\tan\theta > 1$ and $\tan\theta > \cos\theta$ (which doesn't satisfy (*))

rules out B, leaves C and D where their difference is whether $\sin\theta$ is +ve/-ve

If $\sin\theta < 0$, and $\sin\theta > \cos\theta$, $\cos\theta$ is also < 0

but $\tan\theta = \dfrac{\sin\theta}{\cos\theta} = \dfrac{-ve}{-ve}$ = +ve, so $\tan\theta > 0$

$\tan\theta > \sin\theta > \cos\theta$ in this case which doesn't satisfy (*).

Hence $\sin\theta \geq 0$ to satisfy (*), ruling out D and leaving the answer as C.

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G. For all θ in the range $0 \leqslant \theta < 2\pi$ the line

$$(y-1)\cos\theta = (x+1)\sin\theta$$

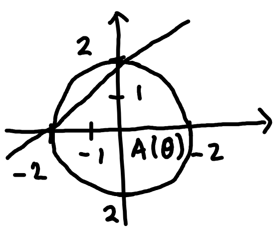
divides the disc $x^2 + y^2 \leqslant 4$ into two regions. Let $A(\theta)$ denote the area of the larger region.

Then $A(\theta)$ achieves its maximum value at

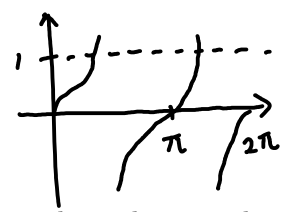
(a) one value of θ (b) two values of θ (c) three values of θ
(d) four values of θ (e) all values of θ.

$(y-1)\cos\theta = (x+1)\sin\theta$

$\tan\theta = \frac{y-1}{x+1}$, so $\tan\theta$ is the gradient of the line through $(-1,1)$ and takes all values.



By inspection, $A(\theta)$ at maximum when $\tan\theta = 1$ = gradient of line, which has 2 solutions in the given range



H. In this question a and b are real numbers, and a is non-zero.
When the polynomial $x^2 - 2ax + a^4$ is divided by $x + b$ the remainder is 1.
The polynomial $bx^2 + x + 1$ has $ax - 1$ as a factor.

It follows that b equals

(a) 1 only (b) 0 or -2 (c) 1 or 2 (d) 1 or 3 (e) -1 or 2.

$f(x) = x^2 - 2ax + a^4$ $g(x) = bx^2 + x + 1$

using factor theorem:

$f(-b) = 1 = b^2 + 2ab + a^4$ (*)

$g\left(\frac{1}{a}\right) = 0 = \frac{b}{a^2} + \frac{1}{a} + 1$

$0 = b + a + a^2$

$b = -a - a^2$, plugging this into (*).

$1 = a^2 - 2a^3 + a^4 + 2a(-a - a^2) + a^4$

$1 = 2a^4 - a^2$

$0 = 2a^4 - a^2 - 1$, let $x = a^2$, $0 = 2x^2 - x - 1$
$= (2x+1)(x-1)$

$x = a^2 = 1$ or $-\frac{1}{2}$

$a = \pm 1$, $b = 0$ or -2

I. Let $a, b, c > 0$ and $a \neq 1$. The equation

$$\log_b\left((b^x)^x\right) + \log_a\left(\frac{c^x}{b^x}\right) + \log_a\left(\frac{1}{b}\right)\log_a(c) = 0$$

has a repeated root when

(a) $b^2 = 4ac$ (b) $b = \frac{1}{a}$ (c) $c = \frac{b}{a}$ ✓(d) $c = \frac{1}{b}$ (e) $a = b = c$.

$\log_b(b^{x^2}) + x\log_a\left(\frac{c}{b}\right) - \log_a(b)\log_a(c) = 0$

$x^2 + x(\log_a c - \log_a b) - \log_a(b)\log_a(c) = 0$

$(x + \log_a c)(x - \log_a b) = 0 \quad (*)$

since roots are the same as the root is repeated

$-\log_a c = \log_a b$

$\log_a\left(\frac{1}{c}\right) = \log_a b$

$b = \frac{1}{c}$

Alternative if you don't spot the factorisation in $(*)$:

set $b^2 - 4ac = 0$ for

$x^2 - x(\log_a c - \log_a b) - \log_a b \log_a c = 0$

$(\log_a(c) - \log_a(b))^2 + 4\log_a(b)\log_a(c) = 0$

$[\log_a(c)]^2 + 2\log_a(b)\log_a(c) + [\log_a(b)]^2 = 0$

$[\log_a(c) + \log_a(b)]^2 = 0$

$\log_a(c) = -\log_a(b)$ as above

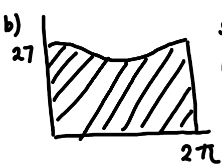
J. Which of these integrals has the largest value? You are not expected to calculate the exact value of any of these.

(a) $\displaystyle\int_0^2 (x^2 - 4)\sin^8(\pi x)\,dx$ (b) $\displaystyle\int_0^{2\pi} (2 + \cos x)^3\,dx$ (c) $\displaystyle\int_0^{\pi} \sin^{100} x\,dx$

✓(d) $\displaystyle\int_0^{\pi} (3 - \sin x)^6\,dx$ (e) $\displaystyle\int_0^{8\pi} 108(\sin^3 x - 1)\,dx$.

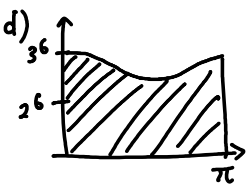
a) (x^2-4) is negative between 2 and 0 so the integral will evaluate as negative

b)



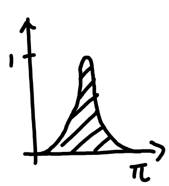
sketch integral. Max value of $(2+\cos x)^3$ occurs at 27 when $\cos x = 1$ Integral < 27 2π as integral shape bounded within rectangle

d)



Minimum value of $(3-\sin x)^6 = 2^6$ when $\sin x = 1$ Hence area must be bigger than $2^6\pi$ rectangle.

c)



area clearly much smaller than b) from sketch

e) $(\sin^3 x - 1)$ is the sin graph translated below the x axis one unit down. Hence integral will evaluate to negative.

Turn over

2. For **ALL APPLICANTS.**

There is a unique real number α that satisfies the equation

$$\alpha^3 + \alpha^2 = 1.$$

[You are not asked to prove this.]

(i) Show that $0 < \alpha < 1$.

(ii) Show that

$$\alpha^4 = -1 + \alpha + \alpha^2.$$

(iii) Four functions of α are given in (a) to (d) below. In a similar manner to part (ii), each is equal to a quadratic expression

$$A + B\alpha + C\alpha^2$$

in α, where A, B, C are integers. (So in (ii) we found $A = -1$, $B = 1$, $C = 1$.) You may assume in each case that the quadratic expression is unique.

In each case below find the quadratic expression in α.

(a) α^{-1}.

(b) The infinite sum

$$1 - \alpha + \alpha^2 - \alpha^3 + \alpha^4 - \alpha^5 + \cdots .$$

(c) $(1 - \alpha)^{-1}$.

(d) The infinite product

$$(1 + \alpha)(1 + \alpha^2)(1 + \alpha^4)(1 + \alpha^8)(1 + \alpha^{16}) \cdots .$$

i) $a^3 + a^2 - 1 = 0$

let $f(a) = a^3 + a^2 - 1$

$f(0) = -1$

$f(1) = 1$

Since sign change and graph continuous between $f(0)$ and $f(1)$, a root 'α' lies between 0 and 1.

Since the question states $f(a)$ has only one real root,

$0 < \alpha < 1$

ii) $a^3 + a^2 = a$

$a^4 + a^3 = a$ and $a^3 = 1 - a^2$

$a^4 = a - a^2 = a - (1 - a^2)$

$a^4 = -1 + a + a^2$

iii) a) $1 = a^3 + a^2$

$a^{-1} = a^2 + a$ $\quad A = 0,\ B = 1,\ C = 1$

b) using the infinite geometric progression formula:

$1 - a + a^2 - a^3 + a^4 \ldots = \frac{1}{1+a} = (1+a)^{-1}$

using part i) $a^3 + a^2 = a^2(1+a) = 1$

$a^2 = (1+a)^{-1}$ $\quad A = 0,\ B = 0,\ C = 1$

c) Assume $(1-a)^{-1} = A + B\alpha + C\alpha^2 = \frac{1}{1-a}$

$1 = (1-a)(A + Ba + Ca^2)$

$1 = A + Ba + Ca^2 - Aa - Ba^2 - Ca^3$

$1 = A + (B-A)a + (C-B)a^2 + C(a^2 - 1)$

$1 = (A-C) + (B-A)a + (2C-B)a^2$

Equating coefficients:

$A - C = 1 \quad B - A = 0 \quad 2C - B = 0$

$A = 1 + C = B = 2C$

$C = 1,\ A = 2,\ B = 2$

d) $(1+a)(1+a^2)(1+a^4)\ldots$

$= 1 + a + a^2 + a^3 + a^4 + a^5 + a^6 \ldots$

$\therefore$ coefficient of all $a^n = 1$ as each power occurs once

$\therefore$ geometric sum to infinite

with $a = 1$ and $r = \alpha$

$= \frac{1}{1-\alpha}$

this is the same as in part c)

$\frac{1}{1-\alpha}$ expressed as $A = 2,\ B = 2,\ C = 1$ as per part iii) c).

Turn over

If you require additional space please use the pages at the end of the booklet

3.

For **APPLICANTS IN** $\left\{\begin{array}{l}\textbf{MATHEMATICS}\\ \textbf{MATHEMATICS \& STATISTICS}\\ \textbf{MATHEMATICS \& PHILOSOPHY}\\ \textbf{MATHEMATICS \& COMPUTER SCIENCE}\end{array}\right\}$ **ONLY.**

Computer Science and *Computer Science & Philosophy* **applicants should turn to page 14.**

For each positive integer k, let $f_k(x) = x^{1/k}$ for $x \geqslant 0$.

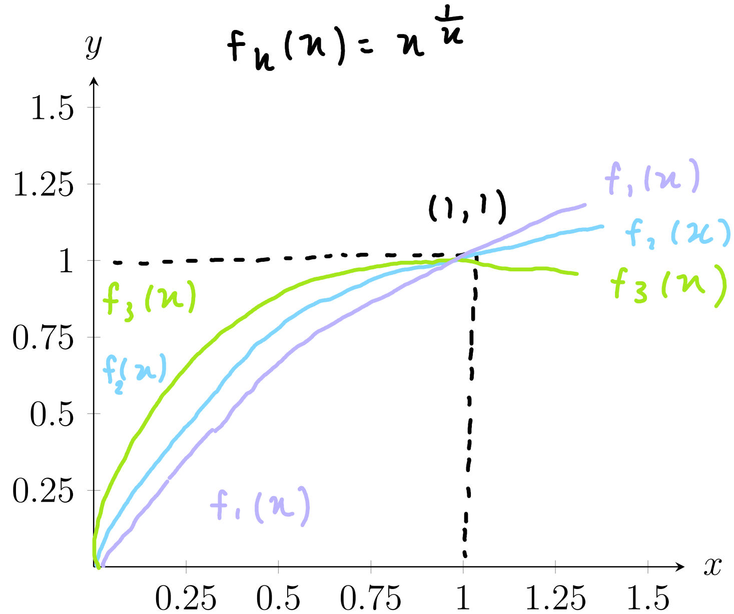
(i) On the same axes (provided below), labelling each curve clearly, sketch $y = f_k(x)$ for $k = 1, 2, 3$, indicating the intersection points.

(ii) Between the two points of intersection in (i), the curves $y = f_k(x)$ enclose several regions. What is the area of the region between $y = f_k(x)$ and $y = f_{k+1}(x)$? Verify that the area of the region between $y = f_1(x)$ and $y = f_2(x)$ is $\frac{1}{6}$.

Let c be a constant where $0 < c < 1$.

(iii) Find the x-coordinates of the points of intersection of the line $y = c$ with $y = f_1(x)$ and of $y = c$ with $y = f_2(x)$.

(iv) The constant c is chosen so that the line $y = c$ divides the region between $y = f_1(x)$ and $y = f_2(x)$ into two regions of equal area. Show that c satisfies the cubic equation $4c^3 - 6c^2 + 1 = 0$. Hence find c.



ii) $\int_0^1 f_{k+1} - f_k = \int_0^1 x^{\frac{1}{k+1}} - x^{\frac{1}{k}}$

$= \left[\frac{k+1}{k+2} x^{\frac{k+2}{k+1}} - \frac{k}{k+1} x^{\frac{k+1}{k}} \right]_0^1$

$= \frac{k+1}{k+2} - \frac{k}{k+1}$

$= \frac{\cancel{k^2} + \cancel{2k} + 1 - \cancel{k^2} - \cancel{2k}}{(k+1)(k+2)}$

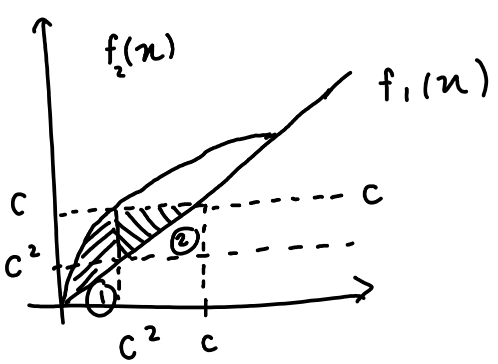
$= \frac{1}{(k+1)(k+2)}$

when $k=1$, region $= \frac{1}{2(3)} = \frac{1}{6}$ as required

iii) At $f_1(x)$, $c = x$ at intersection

At $f_2(x)$, $c = x^{\frac{1}{2}}$, $x = c^2$ at intersection

iv)



split in areas 1 and 2

1 $= \int_0^{c^2} x^{\frac{1}{2}} - x^1 \, dx$

$= \left[\frac{2}{3} x^{\frac{3}{2}} - \frac{1}{2} x^2 \right]_0^{c^2}$

$= \frac{2}{3} c^3 - \frac{1}{2} c^4$

2 = area of triangle $= \frac{1}{2}(c - c^2)(c - c^2)$

$= \frac{1}{2}(c - c^2)^2$

$= \frac{1}{2}\left[c^2 - 2c^3 + c^4\right]$

$= \frac{1}{2}c^2 - c^3 + \frac{1}{2}c^4$

combining 1 and 2 : $\frac{2}{3}c^3 - \frac{1}{2}c^4 + \frac{1}{2}c^2 - c^3 + \frac{1}{2}c^4$

As (total area)/ 2 = 1 + 2 $= \frac{1}{12} = \frac{1}{2}c^2 - \frac{1}{3}c^3$

$1 = 6c^2 - 4c^3$

$\Rightarrow 4c^3 - 6c^2 + 1 = 0$ as required

solve $4c^3 - 6c^2 + 1 = 0$ by inspection to find $c = \frac{1}{2}$

4.

For **APPLICANTS IN** $\left\{\begin{array}{l}\textbf{MATHEMATICS}\\ \textbf{MATHEMATICS \& STATISTICS}\\ \textbf{MATHEMATICS \& PHILOSOPHY}\end{array}\right\}$ **ONLY.**

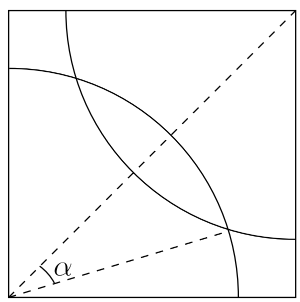
Mathematics & Computer Science, *Computer Science* and *Computer Science & Philosophy* **applicants should turn to page 14.**

A horse is attached by a rope to the corner of a square field of side length 1.

(i) What length of rope allows the horse to reach precisely half the area of the field?

Another horse is placed in the field, attached to the corner diagonally opposite from the first horse. Each horse has a length of rope such that each can reach half the field.

(ii) Explain why the area that both can reach is the same as the area neither can reach.



(iii) The angle α is marked in the diagram above. Show that $\alpha = \cos^{-1}\left(\frac{\sqrt{\pi}}{2}\right)$ and hence show that the area neither can reach is $\frac{4}{\pi}\cos^{-1}\left(\frac{\sqrt{\pi}}{2}\right) - \sqrt{\frac{4-\pi}{\pi}}$. Note that $\cos^{-1}$ can also be written as arccos.

A third horse is placed in the field, and the three horses are rearranged. One horse is now attached to the midpoint of the bottom side of the field, and another horse is now attached to the midpoint of the left side of the field. The third horse is attached to the upper right corner.

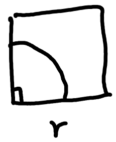
(iv) Given each horse can access an equal area of the field and that none of the areas overlap, what length of rope must each horse have to minimise the area that no horse can reach?

The horses on the bottom and left midpoints of the field are each replaced by a goat; each goat is attached by a rope of length g to the same midpoint as in part (iii). The remaining horse is attached to the upper right corner with rope length h.

(v) Given that $0 \leqslant h \leqslant 1$, and that none of the animals' areas can overlap, show that $\frac{\sqrt{5}-2}{2} \leqslant g \leqslant \frac{1}{2\sqrt{2}}$ holds if the area that the animals can reach is maximised.

i) half area of field $= \frac{1}{2}$

length of rope in field covers a quarter circle as shown in diagram

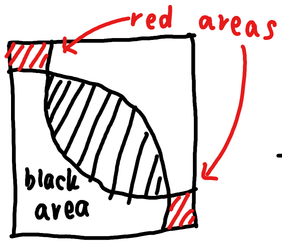


$$\frac{1}{2} = \frac{\pi r^2}{4}$$

$$\frac{2}{\pi} = r^2$$

$$r = \sqrt{\frac{2}{\pi}} = \frac{\sqrt{2}}{\sqrt{\pi}}$$

ii)



The total area covered by the 2 overlapping circles is given yb

$\frac{1}{2} + \frac{1}{2}$ - black area

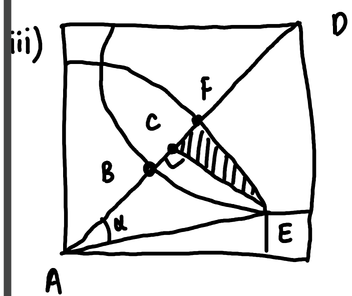
The total area of the square is 1 and given by the red area added to the area covered by the quarter circles.

Hence 1 = red area $+ \frac{1}{2} + \frac{1}{2}$ - black area

0 = red area - black area

black area = red area

iii)



$AD = \sqrt{2}$ as diagonal of square

$AC = \frac{\sqrt{2}}{2}$ as half of AD

$AE = \frac{\sqrt{2}}{\sqrt{\pi}} = r$ from part i)

In ΔACE, $\cos\alpha = \frac{AC}{AE}$

$$\cos\alpha = \frac{\sqrt{2}}{2} \div \frac{\sqrt{2}}{\sqrt{\pi}} = \frac{\sqrt{2}\cdot\sqrt{\pi}}{2\sqrt{2}} = \frac{\sqrt{\pi}}{2}$$

$$\alpha = \arccos\frac{\sqrt{\pi}}{2}$$

Area shaded in red is $\frac{1}{4} \times$ area of total overlap.

Red area = circle segment - triangle area

Areas CEF = AEF - ACE $= \frac{1}{4}$ total overlap

Hence total overlap $= 4[AEF - ACE]$

$$= 4\left[\cancel{\pi}r^2\left(\frac{\alpha}{2\cancel{\pi}}\right) - \frac{1}{2}(AC \times CE)\right]$$

using Pythagoras on ΔACE,

$$(CE)^2 = \left(\frac{\sqrt{2}}{\sqrt{\pi}}\right)^2 - \left(\frac{\sqrt{2}}{2}\right)^2 = \frac{2}{\pi} - \frac{2}{4}$$

$$= \frac{4-\pi}{2\pi}$$

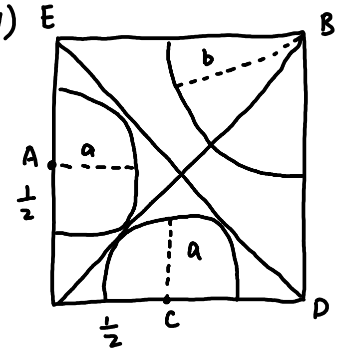
$$CE = \sqrt{\frac{4-\pi}{2\pi}}$$

total overlap

$$= 4\left[\frac{2}{\pi}\cdot\frac{\arccos\left(\frac{\sqrt{\pi}}{2}\right)}{2} - \frac{1}{2}\left(\frac{\cancel{\sqrt{2}}}{2} \times \frac{\sqrt{4-\pi}}{\cancel{\sqrt{2}} \times \sqrt{\pi}}\right)\right]$$

$$= \frac{4}{\pi}\arccos\left(\frac{\sqrt{\pi}}{2}\right) - \sqrt{\frac{4-\pi}{\pi}}$$

4iv)



To minimise area not reached, we need to mamimise area reached by the horses.

As the radii increase, assume that the semicircles meet first.

Areas of semicircles and quarter circle are the same so:

$$\frac{\pi a^2}{2} = \frac{\pi b^2}{4}$$

$$2a^2 = b^2$$

$$a\sqrt{2} = b$$

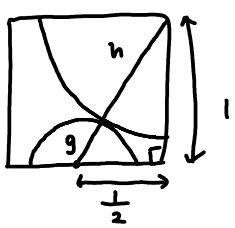
When the semicircles meet, as the horses are attached at the midpoints

diagonal $AC = \frac{\sqrt{2}}{2}$ and $a = \frac{AC}{2} = \frac{\sqrt{2}}{4}$, $b = \frac{2}{4} = \frac{1}{2}$

since $\frac{1}{2} < \frac{\sqrt{2}}{2}$, the quarter circle's radius 'b' has not yet touched the DE diagonal and cannot have touched either of the semicircles, so the initial assumption that the semicircles met first was correct.

iv) From part iv) $g \leq \frac{\sqrt{2}}{4}$ and $g \geqslant 0$

so $0 \leq g \leq \frac{\sqrt{2}}{4}$



Assume when maximising that the quarter circle meets the semicircle first

From the left diagram here,

$$(g+h)^2 = \left(\frac{1}{2}\right)^2 + 1^2 = \frac{5}{4}$$

$$g+h = \sqrt{\frac{5}{4}}$$

$$g = \sqrt{\frac{5}{4}} - h$$

$h_{max} = 1$, so $g_{min} = \frac{\sqrt{5}}{2} - 1 = \frac{\sqrt{5}-2}{2}$

$$g \geqslant \frac{\sqrt{5}-2}{2}$$

hence overall, $\frac{\sqrt{5}-2}{2} \leq g \leq \frac{\sqrt{2}}{4}$

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5. For **ALL APPLICANTS.**

Ten children, $c_0, c_1, c_2, \ldots, c_9$, are seated clockwise in a circle. The teacher walks clockwise behind the children with a large bag of sweets. She gives a sweet to child c_1. She then skips a child and gives a sweet to the next child, c_3. Next she skips two children and gives a sweet to the next child, c_6. She continues in this way, at each stage skipping one more child than at the preceding stage before giving a sweet to the next child.

(i) The kth sweet is given to child c_i. Explain why i is the last digit of the number

$$\frac{k(k+1)}{2}.$$

(ii) Let $1 \leqslant k \leqslant 18$. Explain why the kth and $(20 - k - 1)$th sweets are given to the same child.

(iii) Explain why the kth sweet is given to the same child as the $(k + 20)$th sweet.

(iv) Which children can never receive any sweets?

When the teacher has given out all the sweets, she has walked exactly 183 times round the circle, and given the last sweet to c_0.

(v) How many sweets were there initially?

(vi) Which children received the most sweets and how many did they receive?

i) $C_0 \quad C_1 \quad C_2 \quad C_3 \quad C_4 \quad C_5 \quad C_6 \quad C_7 \quad C_8 \quad C_9$

1^{st} sweet given to C_1

2^{nd}: $1 +$ skip $1 + 1 = 1 + 2 \Rightarrow C_3$

3^{rd}: $1 + 2 +$ skip $2 + 1 = 1 + 2 + 3 = C_6$

n^{th} sweet given to $C_{(1+2+3+\ldots n)} = C\left(\sum_{i=1}^{n} i\right)$

$=$ sum of natural numbers

$= \frac{n(n+1)}{2} \rightarrow$ number of steps until n^{th} sweet

we only pay attention to the last digit

(as the child number is the last digit)

ii) plugging in 'k' and then '20-k-1' into the formula in part i):

$\frac{k(k+1)}{2}$ and $\frac{(20-k-1)(20-k)}{2}$

$$\frac{(20-k-1)(20-k)}{2} = \frac{20(20-k-1)}{2} + \frac{k(k+1-20)}{2}$$
$$= 10(20-k-1) - 10k + \frac{k(k+1)}{2}$$
$$= 10(20-2k-1) + \frac{k(k+1)}{2}$$

since the formula for '20-k-1' is of the form $10n + \frac{k(k+1)}{2}$

it will have the same units digits as $\frac{k(k+1)}{2}$, hence 'i' will be the same for both 'k' and '20-k-1'

iii) using part ii), the 18^{th} sweet is given to the same child as the 1^{st} sweet, who was C_1.

19^{th} sweet to C_0

20^{th} sweet to C_0

21^{st} sweet to C_1 – same as 1st sweet

The pattern repeats as it starts again from C_1, and that 'k' and '$10\times2+k$' produce the same 'i'.

iv) From part ii), sweets 1-9 inclusive are given to the same children as 18-10 inclusive.

From part iii), sweets 19 and 20 are given to C_0.

Since the pattern repeats every 20 sweets, it is only necessary to consider the first 20 sweets.

Listing out the first 9 numbers of the form $\frac{k(k+1)}{2}$: (the triangle numbers)

1, 3, 6, 10, 15, 21, 28, 36, 45 ..

It can be seen that 2, 4, 7 and 9 do not appear as 'i' in the units digit.

Hence C_2, C_4, C_7 and C_9 never receive any sweets sadly

v) 183 circles $\times$ 10 = 1830

$$\frac{k(k+1)}{2} = 1830$$

$$k^2 + k - 3660 = 0$$
$$(k+61)(k-60) = 0$$

$k = 60$ sweets initially

vi) using our list from part iv), we can count how many sweets each child receives per every 20 sweets handed out in total

child	1	3	5	6	8	0
sweets	4	2	4	4	2	4

children 0, 1, 5 and 6 receive the most sweets

since 60 sweets were handed out, $\frac{60}{20} = 3$ repeats

so each child receiving the most will get $4 \times 3 = 12$ sweets

6.

For **APPLICANTS IN** $\left\{ \begin{array}{l} \textbf{COMPUTER SCIENCE} \\ \textbf{MATHEMATICS \& COMPUTER SCIENCE} \\ \textbf{COMPUTER SCIENCE \& PHILOSOPHY} \end{array} \right\}$ **ONLY.**

You need to pack several items into your shopping bag, without squashing any item. Suppose each item i has a (positive) weight w_i, and a strength s_i which is the maximum weight that can be placed above it without it being squashed. For the purposes of this question, suppose that the items will be arranged one on top of the other within your bag. We will say that a particular packing order is *safe* if no item is squashed, that is, for each item i, s_i is at least the sum of the w_j corresponding to items j placed above item i. For example, suppose we have the following items, packed in the order given

Ordering	Item	w_i	s_i
Top	Apples	5	6
Middle	Bread	4	4
Bottom	Carrots	12	9

This packing is not safe: the bread is squashed, since the weight above it (5) is greater than its strength (4). However, swapping the apples and the bread gives a safe packing.

(i) Which of the other four orderings of apples, bread, and carrots are safe or unsafe?

(ii) Consider the tactic of packing the items in weight order, with the heaviest at the bottom. Show by giving an example that this might not produce a safe packing order, even if a safe packing order exists.

(iii) Now consider the tactic of packing the items in strength order, with the strongest at the bottom. Again show by giving an example that this might not produce a safe packing order, even if one exists.

(iv) Suppose we have a safe packing order, with item j *directly* on top of item i. Suppose further that

$$w_j - s_i \;\geqslant\; w_i - s_j.$$

Show that if we swap items i and j, we still have a safe packing order.

(v) Hence suggest a practical method of producing a safe packing order if one exists. Explain why your method works. (Listing all possible orderings is not practical.)

i) All other orderings are unsafe as carrots would be on top or in the middle, and neither the apples not the bread have the strength to support the carrots' weight of 12.

ii) Any suitable example works, e.g

	Item	w_i	s_i
top	x	5	11
bottom	y	10	4

Here 'y' is heavier but produces an unsafe ordering, whereas swapping the lighter 'x' to the bottom is safe

iii) e.g.

	Item	w_i	s_i
top	x	11	5
bottom	y	4	10

$s_y < w_x$ so unsafe
but if 'x' swapped to bottom, $s_x > w_y$ so safe

iv)

top	j	w_j	s_j
bottom	i	w_i	s_i

And let weight of all items above 'i' and 'j' be 'w_a'
As the current ordering is safe,

$$s_i \geq w_j + w_a$$
$$s_i - w_j \geq w_a$$

And it is given that: $w_j - s_i \geq w_i - s_j$

$$s_j - w_i \geq s_i - w_j \geq w_a$$
$$s_j - w_i \geq w_a$$
$$s_j \geq w_i + w_a$$

Hence 'j' can support the weight of 'i' and all other items above so ordering still safe if 'i' and 'j' are swapped.

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6v) It has been shown that considering weight or strength alone is not sufficient part iv) can be rearranged to:

$$w_j + s_j \geqslant w_i + s_i \quad \text{for a safe swap}$$

So for a safe packing order, both weight and strength need to be considered, with the items where '$w_i + s_i$' together is largest should be placed at the bottom.

if there exists a safe packing order, adjacent items on the pile must be swapped to move items with the largest '$w_i + s_i$' on the bottom until the items are in the correct order of '$w_i + s_i$' magnitude.

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7.

For **APPLICANTS IN** $\left\{ \begin{array}{l} \textbf{COMPUTER SCIENCE} \\ \textbf{COMPUTER SCIENCE \& PHILOSOPHY} \end{array} \right\}$ **ONLY.**

A simple computer can operate on lists of numbers in several ways.

- Given two lists a and b, it can make the *join* $a + b$, by placing list b after list a. For example if

$$a = (1, 2, 3, 4) \quad \text{and} \quad b = (5, 6, 7) \quad \text{then} \quad a + b = (1, 2, 3, 4, 5, 6, 7).$$

- Given a list a it can form the *reverse* sequence $R(a)$ by listing a in reverse order. For example if

$$a = (1, 2, 3, 4) \quad \text{then} \quad R(a) = (4, 3, 2, 1).$$

(i) Given sequences a and b, express $R(a + b)$ as the join of two sequences. What is $R(R(a))$?

- Given a sequence a of length n and $0 \leqslant k \leqslant n$, then the kth *shuffle* S_k of a moves the first k elements of a to the end of the sequence in reverse order. For example

$$S_2(1, 2, 3, 4, 5) = (3, 4, 5, 2, 1) \quad \text{and} \quad S_3(1, 2, 3, 4, 5) = (4, 5, 3, 2, 1).$$

(ii) Given two sequences a and b, both of length k, express $S_k(a + b)$ as the join of two sequences. What is $S_k(S_k(a + b))$?

(iii) Now let $a = (1, 2, 3, 4, 5, 6, 7, 8)$. Write down

$$S_5(S_5(a))$$

as the join of three sequences that are either in order or in reverse order. Show that the sequence a is back in its original order after four S_5 shuffles.

(iv) Now let a be a sequence of length n with $k \geqslant n/2$. Prove, after S_k is performed four times, that the sequence returns to its original order.

(v) Give an example to show that when $k < n/2$, the sequence need not be in its original order after S_k is performed four times. For your example how many times must S_k be performed to first return the sequence to its original order?

End of last question

i) $R(a+b) = R(b) + R(a)$

$R(R(a)) = a$

ii) As both A and B length k,

$S_k(a+b) = b + R(a)$

$S_k(S_k(a+b)) = R(a) + R(b)$ [= since $S_k(b+R(a))$]

iii) $S_5(a) = (6,7,8,5,4,3,2,1)$

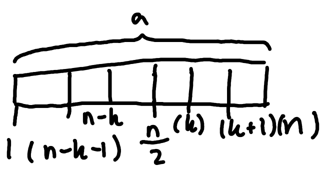
$S_5(S_5(a)) = S_5(6,7,8,5,4,3,2,1) = (3,2,1,4,5,8,7,6)$

$= (3,2,1) + (4,5) + (8,7,6)$

$S_5(3,2,1,4,5,8,7,6) = (8,7,6,5,4,1,2,3)$

$S_5(8,7,6,5,4,1,2,3) = (1,2,3,4,5,6,7,8) = a$

iv) For a general list 'a':



The correct order of critical values is:

$1, \ldots, n-k-1, n-k, \frac{n}{2} \ldots k, | k+1, \ldots n$

let the red line mark the first k values in the list.

After the 1st shuffle, the list becomes:

$k+1, \ldots n, k, \ldots, \frac{n}{2}, \ldots, n-k, k, | n-k-1, \ldots, 1$

2nd shuffle:

$n-k-1, \ldots, 1, n-k, \ldots, \frac{n}{2}, \ldots, k, | n, \ldots, k+1$

3rd shuffle:

$n, \ldots, k+1, k, \ldots, \frac{n}{2}, \ldots, n-k, | 1, \ldots, n-k-1$

4th shuffle:

$1, \ldots, n-k-1, n-k, \ldots, \frac{n}{2} \ldots, k, k+1, \ldots, n$

where it can be seen that the critical values are in the same order as the initial ordering of 'a'.

Since the values between these critical values have not been used as a point of shuffling at the k^{th} value of a list, they have remained in the same relative positions to their immediate critical points on both sides.

Hence the 4th shuffle list is the same as the initial list, as all the critical and non-critical points are in the same positions.

Turn over

If you require additional space please use the pages at the end of the booklet

v) $a = (1,2,3)$ so $n=3$, let $k=1$ so that $k < \frac{n}{2}$

1st : [2, 3, 1]

2nd : [3, 1, 2]

3rd : [1, 2, 3]

4th : [2, 3, 1]

For this list, 3 iterations of S_k returns it to the original position.

The 4th does not return this list to its original.

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